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$$y = -2\sqrt{-a} \sin \frac{1}{5} \left(2\pi + \sin^{-1} \frac{b}{a^2 \sqrt{-a}} \right),$$

$$y = -2\sqrt{-a} \sin \frac{1}{5} \left(\pi - \sin^{-1} \frac{b}{a^2 \sqrt{-a}} \right),$$

$$y = 2\sqrt{-a} \sin \frac{1}{5} \left(\pi + \sin^{-1} \frac{b}{a^2 \sqrt{-a}} \right),$$

$$y = 2\sqrt{-a} \sin \frac{1}{5} \left(2\pi - \sin^{-1} \frac{b}{a^2 \sqrt{-a}} \right).$$

2. The equation

$$xy\sqrt{-(b^2 + a^5)} = a^2y^3 + by^2 + 3a^3y + 2ab \quad (1)$$

represents a real trident of Newton¹ when $b^2 + a^5$ is negative, and

$$x^2 + y^2 = -4a \quad (2)$$

is the equation of a circle, which will be real when a is negative. Eliminating x from (1) by means of (2), the result may be placed under the form

$$(y^5 + 5ay^3 + 5a^2y + 2b)(a^2y + 2b) = 0.$$

The first of these factors is identical with the quintic expression given above and has five real roots when $b^2 + a^5$ is negative. From this it appears the five real roots may be represented by the ordinates of the intersections of a trident and a circle. The five intersections, whose ordinates are the five real roots of this equation, are the vertices of a regular pentagon inscribed in a circle whose radius is $2\sqrt{-a}$, as may be seen by reference to the above values of y .

II. CERTAIN MATHEMATICAL FEATURES OF THERMODYNAMICS.

By J. E. TREVOR, Cornell University.

Let e , v , s denote the energy, volume, and entropy of unit mass of a body of homogeneous fluid in a state of thermodynamic equilibrium under the pressure p at the temperature θ . The energy e is then a continuous function of v , s , and equilibrium subsists when and only when

$$(1) \quad de = -pdv + \theta ds.$$

Hence the conditions of equilibrium are the equations

$$(2) \quad -p = \partial e / \partial v, \quad \theta = \partial e / \partial s.$$

The state of equilibrium is stable, with respect to small displacements, when

¹ One of the four canonical forms (no. 108) to which Newton has reduced the general equation of cubic curves in his *Enumeratio linearum tertii ordinis*, first printed in Newton's *Optics*, London, 1704.—EDITOR.

and only when all possible sets of variations δe , δv , δs of the energy, volume, and entropy of the body satisfy the inequality

$$\delta e + p\delta v - \theta\delta s > 0.$$

Hence the necessary and sufficient conditions of stability are

$$\frac{\partial^2 e}{\partial v^2} \frac{\partial^2 e}{\partial s^2} - \left(\frac{\partial^2 e}{\partial v \partial s} \right)^2 > 0, \quad \frac{\partial^2 e}{\partial v^2} > 0.$$

A consequence is that $\partial^2 e / \partial s^2 > 0$.

For various purposes it is desirable to employ v , θ or p , s or p , θ as independent variables, the remaining three of the variables e , p , v , θ , s then being functions of those chosen. The determinants of these transformations are

$$\frac{\partial(v, \theta)}{\partial(v, s)} = \frac{\partial^2 e}{\partial s^2}, \quad \frac{\partial(p, s)}{\partial(v, s)} = -\frac{\partial^2 e}{\partial v^2}, \quad \frac{\partial(p, \theta)}{\partial(v, s)} = -\frac{\partial^2 e}{\partial v^2} \frac{\partial^2 e}{\partial s^2} + \left(\frac{\partial^2 e}{\partial v \partial s} \right)^2,$$

and by the conditions of stability none of these determinants vanish. When functions of the new variables are defined by the equations

$$f(v, \theta) = e - \theta s, \quad g(p, s) = e + pv, \quad h(p, \theta) = e + pv - \theta s,$$

differentiation and comparison with (1) yields the criterion of equilibrium in the forms

$$(3) \quad df = -p dv - s d\theta, \quad dg = v dp + \theta ds, \quad dh = v dp - s d\theta.$$

The conditions of integrability of the second members of (1) and (3) are Maxwell's "thermodynamic relations,"

$$(4) \quad -\frac{\partial p}{\partial s} = \frac{\partial \theta}{\partial v}, \quad \frac{\partial p}{\partial \theta} = \frac{\partial s}{\partial v}, \quad \frac{\partial v}{\partial s} = \frac{\partial \theta}{\partial p}, \quad \frac{\partial v}{\partial \theta} = -\frac{\partial s}{\partial p}.$$

It may be asked whether p , v and θ , s can serve as sets of independent variables. We have that

$$-\frac{\partial(v, p)}{\partial(v, s)} = \frac{\partial(\theta, s)}{\partial(v, s)} = \frac{\partial^2 e}{\partial v \partial s};$$

and, as discussed below, it may be taken to be a fact of observation that these determinants vanish, if at all, only at points on a line $\partial^2 e / \partial v^2 = 0$. The circumstance that any one of the variables p , v , θ , s is a function of any two of the others gives rise to the consideration of six pairs of variables that can be taken successively independent, of twelve functions of pairs of variables, and of a set of twenty-four first derivatives of these functions. The derivatives of the set may conveniently be assigned to two classes, Class I containing the derivatives of the "work variables" p , v , and of the "heat variables" θ , s , with regard to each other, and Class II containing the derivatives of a work variable with regard to a heat variable, and the reverse. Making use of the thermodynamic relations (4)

connecting the cross derivatives, the proposed classification is exhibited in the following table:

Class I	Class II
$-\left(\frac{\partial p}{\partial v}\right)_\theta = \frac{\Delta}{e_{22}}$	$-\left(\frac{\partial p}{\partial s}\right)_v + \left(\frac{\partial \theta}{\partial v}\right)_s = \frac{e_{12}}{1} = e_{12}$
$-\left(\frac{\partial p}{\partial v}\right)_s = \frac{e_{11}}{1}$	$-\left(\frac{\partial p}{\partial \theta}\right)_v - \left(\frac{\partial s}{\partial v}\right)_\theta = \frac{e_{12}}{e_{22}} = f_{12}$
$+\left(\frac{\partial \theta}{\partial s}\right)_p = \frac{\Delta}{e_{11}}$	$-\left(\frac{\partial v}{\partial s}\right)_p - \left(\frac{\partial \theta}{\partial p}\right)_s = \frac{e_{12}}{e_{11}} = -g_{12}$
$+\left(\frac{\partial \theta}{\partial s}\right)_v = \frac{e_{22}}{1}$	$-\left(\frac{\partial v}{\partial \theta}\right)_p + \left(\frac{\partial s}{\partial p}\right)_\theta = \frac{e_{12}}{\Delta} = -h_{12}$

Using the notation $e_{11} = \partial^2 e / \partial v^2$, $e_{12} = \partial^2 e / \partial v \partial s$, $e_{22} = \partial^2 e / \partial s^2$, $\Delta = e_{11}e_{22} - e_{12}^2$, the tabulated values of the derivatives in terms of second derivatives of $e(v, s)$ are found from the equations

$$-dp = e_{11}dv + e_{12}ds, \quad d\theta = e_{12}dv + e_{22}ds,$$

obtained by differentiating the conditions of equilibrium (2). The derivatives of Class II have the values e_{12} , f_{12} , $-g_{12}$, $-h_{12}$; *i.e.*, they form the terms of the conditions of integrability of the second members of the equations (1) and (3). The tabulated derivatives and their reciprocals constitute the set of twenty-four derivatives. By the "reciprocal" of $(\partial p / \partial v)_\theta = -\Delta / e_{22}$, for example, is meant the derivative $(\partial v / \partial p)_\theta = -e_{22} / \Delta$.

The signs of the twenty-four rates of change of the quantities p , v , θ , s with regard to one another afford extensive information concerning the thermodynamic properties of fluids. By the conditions of stability we have that, for all realizable states of equilibrium, e_{11} , e_{22} , Δ are positive. It appears then that, for realizable states of equilibrium, the signs of the derivatives of Class I are determined by the conditions of stability, while the signs of the derivatives of Class II are determined by the sign of e_{12} , which must be found by observation. From $(\partial v / \partial \theta)_p = -e_{12} / \Delta$ we observe that e_{12} is negative when the specific volume rises on heating at constant pressure, and that e_{12} changes sign if the representative point crosses a line of maximum density at constant pressure. It is clear that the fluid states ordinarily observed lie in a field A (in which $e_{12} < 0$), but that a field B (in which $e_{12} > 0$) appears if, as in the case with liquid water, a locus of maximum density occurs. The derivatives whose values (having regard to their signs) are e_{12} , f_{12} , $-g_{12}$, $-h_{12}$ are negative in the field A , positive in the field B , and zero on the "zero line" $e_{12} = 0$. A moment's reflection serves to refer a given derivative, with the proper sign, to e_{12} or f_{12} , etc. When "heating" is understood to mean increase of entropy or rise of temperature, and "expansion" means increase of volume or fall of pressure, the circumstance that e_{12} is positive

at all points in the field B requires that, in this field, *heating at constant volume decreases the pressure, and heating at constant pressure decreases the volume, while isentropic expansion increases the temperature, and isothermal expansion develops heat.* In the field A the contrary occurs.

Many interesting conclusions may be drawn. For example, by the definitions of the specific heats at constant volume and at constant pressure, and by reference to the table,

$$c_p = \theta \left(\frac{\partial s}{\partial \theta} \right)_p = \theta \frac{e_{11}}{\Delta}, \quad c_v = \theta \left(\frac{\partial s}{\partial \theta} \right)_v = \frac{\theta}{e_{22}};$$

whence, by subtraction,

$$\frac{c_p - c_v}{\theta} = \frac{e_{11}}{\Delta} - \frac{1}{e_{22}} = \frac{e_{12}^2}{e_{22}\Delta}.$$

It follows that, $c_p > c_v$ everywhere except at points on the zero line, where $c_p = c_v$.

Again, choosing v , θ as independent variables, let us consider the specific heat c of a fluid with regard to an arbitrary assigned path $\phi(v, \theta) = 0$. The "heat-differential" θds is

$$\theta \frac{\partial s}{\partial \theta} d\theta + \theta \frac{\partial s}{\partial v} dv;$$

wherefore, on dividing by $d\theta$ and using the second thermodynamic relation and the definition of c_v ,

$$c = c_v + \theta \frac{\partial p}{\partial \theta} \frac{dv}{d\theta},$$

where $dv/d\theta$ is the slope of the path. The quantity $\partial p/\partial \theta$, which is equal to $-f_{12}$, is positive in the field A and negative in the field B . Hence at any point in A the specific heat c is an increasing linear function of $dv/d\theta$, while at any point in B it is a decreasing linear function of this slope. Its value at a given point depends only on the *slope* of the path. At any point on the zero line $\partial p/\partial \theta = 0$, wherefore c is *constant* for all finite values of $dv/d\theta$. In particular $c_p = c_v$, as found above. When $dv/d\theta$ is infinite the value of c must be found by evaluating an indeterminate form. Two paths for which $dv/d\theta$ is infinite are the isotherm, for which $c = \infty$, and the isentropic, for which $c = 0$.

A graphic representation of the values of the general specific heat at points on the zero line is had when paths through the point are drawn on the surface representing the region of fluid states in the v , s , θ - space. In this surface the zero line is the locus of the minimum points of isentropic section of a trough. For at points on the zero line $(\partial \theta/\partial v)_s = e_{12} = 0$.

When the path is the curve of section of the surface by a cylinder (or plane) perpendicular to the s , θ -plane and cutting the zero line, its projection on the v , θ -plane has a horizontal tangent at the zero line. Its projection on the s , θ -plane is a curve with two coincident branches, whose slope $ds/d\theta$ at the point of meeting the zero line determines the value, at this point, of the specific heat $c = \theta \cdot ds/d\theta$ with regard to the path. The isotherm, for which $ds/d\theta = \infty$, and the isentropic, for which $ds/d\theta = 0$, are such paths.

When the path is any other curve whose projection on the v, θ -plane has a horizontal tangent at the zero line, its projection on the s, θ -plane is a curve of two distinct branches that meet the zero line in a cusp. The common slope $ds/d\theta$ of these branches at the point of meeting determines the value, at this point, of the specific heat c with regard to the path.

When the path is any curve whose projection on the v, θ -plane has not a horizontal tangent at the point of meeting the zero line, its projection on the s, θ -plane is a curve tangent to the zero line at the point of meeting. Hence the slope $ds/d\theta$ of the zero line at this point determines the value there of the specific heat c with regard to the path. The isobar and the isometric (the curve of constant volume) through the point are such paths.

III. A CHECK FORMULA FOR THE AMBIGUOUS CASE IN PLANE TRIANGLES.

By W. R. RANSOM, Tufts College.

In the solution of a triangle for which sides a and b and angle A are given, two values B' and B'' are first obtained for the angle opposite b ; then two angles C' and C'' are found, and finally two sides c' and c'' . The obvious relation $\frac{1}{2}(c' + c'') = b \cos A$ may be used to discover the presence of an error in either of the two triangles that have been computed. This formula does not appear in any text book with which I am acquainted: has it not been employed by some one?

IV. THE "KING'S CHAMBER" AND THE GEOMETRY OF THE SPHERE.

By F. J. DICK, Râja-Yoga College.

That the designers of the Great Pyramid possessed a thorough knowledge of the geometry of the sphere has been recognized by some, although the usual view¹ is confined to the recognition of their knowledge of the value of π . The length of the "King's Chamber" is exactly double the breadth, while its height is exactly half the diagonal of the floor.² Thus if the width be called 2, and the length 4, the "cubic diagonal" of the chamber is 5.

Attention is drawn to the significance of this in connexion with the geometry of the sphere. Let the rectangle $DABC$, 4 units by 2, represent the floor plan (a shape, by the way, found in many ancient temples). Let the circumscribed circle represent the diametral section of a sphere and let two other spheres touch at the center as shown forming the double *vesica piscis* $\phi O \eta S$ and $\epsilon N \theta O$, determining the planes JH and FG cutting the cylindrical envelope $\kappa \lambda \mu \nu$. The diagonals JG and FH coincide with the diagonals of $DABC$, and are each 5 in length, *i.e.*, they are of the length of the cubic diagonal of the "King's Chamber." These are the traces of cones cutting the sphere $WNES$ in the small circles whose diameters are AB, CD . Join AN , and draw the circle $TMPQR$. Then AN measures a side of the pentagon $TMPQR$ whose diagonals only are shown. Pro-

¹ W. M. F. Petrie, *The Pyramids and Temples of Gizeh*, London, 1883.

² *Op. cit.*, p. 195.